MATH 347: FUNDAMENTAL MATHEMATICS, FALL 2015

HOMEWORK 7

Due date: Oct 21 (Wed)

Exercises from the textbook. No exercises from the textbook.

Out-of-the-textbook exercises (these are as mandatory as the ones from the textbook)**.**

Definition. For sets A, B, we say that A *injects into* B, and denote this by $A \hookrightarrow B$, if there is an injection $f : A \to B$.

1. Let A be a set and let $f : A \to [n+1]$ be an injection. Prove that if f is not surjective, then there is an injection $g : A \to [n]$.

HINT: Consider the cases depending on whether $n+1 \in f(A)$ or not. In each case, define g using f. In the case when $n+1 \in f(A)$, keep in mind that f is not surjective.

2. Prove that for all finite sets A, B, the following holds:

(a) $|A| \leq |B|$ if and only if $A \hookrightarrow B$.

(b) $|A| < |B|$ if and only if $A \hookrightarrow B$ but there is no bijection between A and B.

3. (a) Let A, B be finite sets. Prove that if there is a non-surjective injection $f : A \to B$, then $|A| < |B|$.

HINT: Let $n = |A|, m = |B|$, and define a non-surjective injection of $[n]$ into $[m]$. Now use Exercise 1 above.

(b) Conclude that if a set D is in bijection with its *proper* subset, then D must be infinite.

HINT: Prove the contrapositive using part (a).

REMARK: This is Exercise 4.43 of the textbook.

4. Let A, B be sets.

(a) Prove that for any surjection $f : A \to B$ there is an injection $g : B \to A$ such that $f \circ g = \mathrm{id}_B$, i.e. for every $b \in B$, $f(g(b)) = b$.

REMARK: This g is not necessarily the inverse of f because it may not be true that $g \circ f = \mathrm{id}_A$. However, it's called a *right-inverse* of f.

HINT: For every $b \in B$, $g(b)$ has to be an element of $I_f(b)$.

(b) Assuming that A, B are finite, prove that if there is a surjection $f : A \to B$, then $|A| \geq |B|$.

5. Let A be a finite set and $f : A \to A$. Prove that f is injective if and only if it is surjective.

HINT: First prove the forward direction $\Rightarrow$ using from Exercise 2a. To prove the backward direction $\Leftarrow$, use Exercise 4a (above) in tandem with the forward direction of the current equivalence that you have already proven.

REMARK: This is Exercise 4.45 of the textbook.